2017-10-18

EE362

Week #4

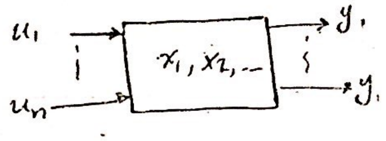
Tiny

Modeling in the Time Domain...........................EE362

- As systems become more complex, representing them with differential equations or transfer functions becomes cumbersome. This is even truer if the system has multiple inputs and outputs. This lecture introduces the ***state space method*** which largely alleviates this problem.

In control engineering, a state space representation is a mathematical model of a physical system as a set of input, output and state variables related by first-order differential equations

The state space representation of a system replaces an **nth** order differential equation with a single **first order *matrix*** differential equation. For any system such as electrical or dynamical system, there exist a set of variables **(states)** such that if the values of the states are known an any reference time t0and if the input is known for t ≥t0then the output and the states can be determined for t ≥t0.



State variable model

1- The State Equations

In the standard form the mathematical description of the system is expressed as a set of n coupled ***first-order ordinary differential equations***, known as the ***state equations***, in which the time derivative of each state variable is expressed in terms of the state variables x1(t), . . . , xn(t) and the system inputs u1(t), . . . , ur(t).

In the general case the form of the n state equations is:

$$\begin{aligned} \dot{x}_1 &= f_1(\mathrm{x}, \mathrm{u}) \\ \dot{x}_2 &= f_2(\mathrm{x}, \mathrm{u}) \\ \vdots &= \vdots \\ \dot{x}_n &= f_n(\mathrm{x}, \mathrm{u}) \end{aligned}$$

Too much

For an LTI system of order n, and with r inputs, Eqs. (1) become a set of n coupled first-order linear differential equations with constant coefficients:

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + b_{11}u_1 + \ldots + b_{1r}u_r \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + b_{21}u_1 + \ldots + b_{2r}u_r \\ &\vdots \\ \dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + b_{n1}u_1 + \ldots + b_{nr}u_r \end{aligned}$$

where the coefficients aij and bij are constants that describe the system.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1r} \\ b_{21} & & b_{2r} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nr} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix}$$

which may be summarized as:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

where the state vector x is a column vector of length n, the input vector u is a column vector of length r, **A is an n × n square matrix** of the constant coefficients aij, and **B is an n × r** matrix of the coefficients bij that weight the inputs.

2- Output Equations

An arbitrary output variable in a system of order n with r inputs may be written:

$$y(t) = c_1x_1 + c_2x_2 + \ldots + c_nx_n + d_1u_1 + \ldots + d_ru_r$$

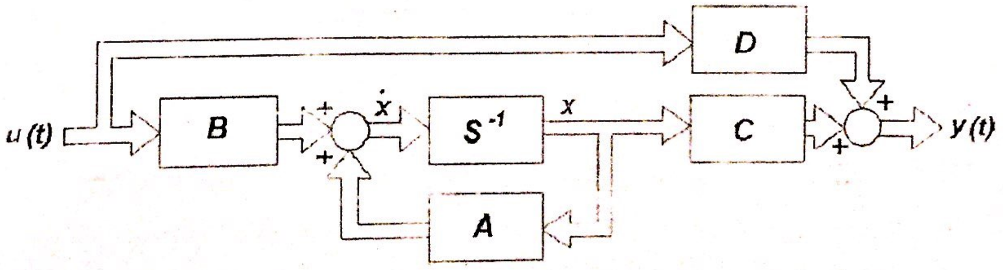
where the ci and di are constants. If a total of m system variables are defined as outputs, the m such equations may be written as:

$$\begin{array}{lcccccccccccccc} y_1 & = & c_{11}x_1 & + & c_{12}x_2 & + & \dots & + & c_{1n}x_n & + & d_{11}u_1 & + & \dots & + & d_{1r}u_r \\ y_2 & = & c_{21}x_1 & + & c_{22}x_2 & + & \dots & + & c_{2n}x_n & + & d_{21}u_1 & + & \dots & + & d_{2r}u_r \\ \vdots & & \vdots & & & & & & & & & & & & \\ y_m & = & c_{m1}x_1 & + & c_{m2}x_2 & + & \dots & + & c_{mn}x_n & + & d_{m1}u_1 & + & \dots & + & d_{mr}u_r \end{array}$$

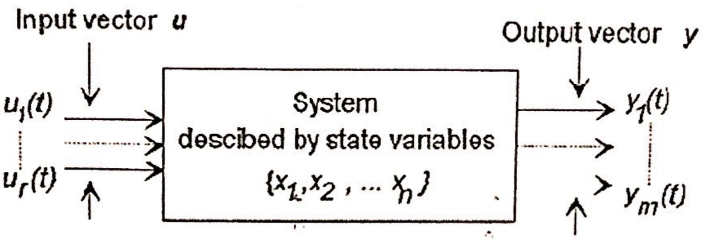
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

where y is a column vector of the output variables yi(t), **C is an m × n** matrix of the constant coefficients cij that weight the state variables, and D is an **m × r** matrix of the constant coefficients dij that weight the system inputs. For many physical systems the matrix D is the null matrix, and the output equation reduces to a simple weighted combination of the state variables:

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$



Summary: A system is represented in state space by the following equations:



$$\boxed{\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}}$$

$$\boxed{\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}}$$

for t > to and initial conditions, x(t0), where

x = state vector $(n \times 1)$

$\dot{\text{x}}$ = derivative of the state vector with respect to time $(n \times 1)$

y = output vector $(m \times 1)$

u = input or control vector $(r \times 1)$

A = system matrix $(n \times n)$

B = input matrix $(n \times r)$

C = output matrix $(m \times n)$

D = feedforward matrix $(m \times r)$

State variables: the smallest set of linearly independent system variables such that the values of the set members at time t along with known forcing functions completely determine the value of all system variables for all t=t0.

State vector: vector whose elements are the state variables.

State space: The n-dimensional space whose bases are the state variables

State equations: set of n simultaneous, first order differential equations with n variables, where the n variables to be solved are the state variables

Output equation: algebraic expression of the output variables of a system as linear combinations of the state variables and the inputs.

Selecting the state variables must follow these rules:

1- A minimum number of state variables must be selected.

The minimum number of state variables required equals the **order of the differential equation** describing the system.

From the transfer: The order of the differential equation is the order of the denominator of the TF after canceling common factors in the numerator and denominator.

A practical way to determine the number of state variables is to count the number of independent energy-storage elements in the system
(capacitors and inductors in electrical syst.; masses and springs in mechanical syst.).

2- They must be linearly independent.

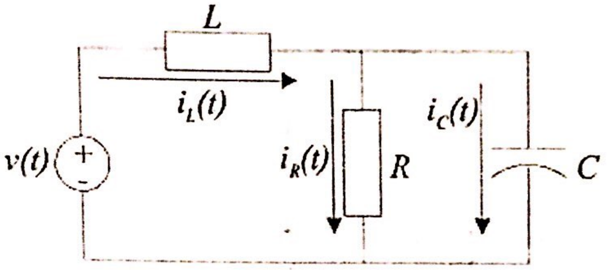
Q) What is meant by linear independence ?
No variable in the set can be written as a linear sum of the other variables in the set.

Recommended procedure:

- Write the simple derivative **eq** for each energy-storage element.
- Solve for each derivative term as a linear combination of system variables and the input.
- Each differentiated variable is selected as a state variable.
- All other system variables and the output are written in terms of the state variables and the input.

Note:-
A state space representation is not unique , since a different choice of state variables leads to different representation of the same system.

Ex: Find a state-space representation of the system shown in the figure if the output is the current through the resistor.



Step 1 Label all the branch currents in the network: i_c, i_r, i_l

Step 2 Select the state variables: write the derivative equation for all energy-storage elements (L and C) and choose the differentiated quantities as the state variables (V_c, i_l)

$$\boxed{C\frac{dv_c}{dt} = i_C \qquad L\frac{di_L}{dt} = v_L}$$

Step 3 Write the right-had sides of "step 2" and the output equations as linear combinations of the state variables and the input. Network theory (Kirchhoff, etc.) will be used for the electric systems.

$$\boxed{\begin{aligned} C\frac{dv_c}{dt} &= i_C = f_1(v_C,\, i_L,\, v(t)) \\ L\frac{di_L}{dt} &= v_L = f_2(v_C,\, i_L,\, v(t)) \end{aligned}}$$

$$\boxed{i_C = -i_R + i_L = -\tfrac{1}{R}v_C + i_l}$$

$$\boxed{v_L = -v_C + v(t)}$$

State eq
$$\boxed{\begin{aligned} \frac{dv_c}{dt} &= -\tfrac{1}{RC}v_C + \tfrac{1}{C}i_L \\ \frac{di_L}{dt} &= -\tfrac{1}{L}v_C + \tfrac{1}{L}v(t) \end{aligned}}$$

Output eq
$$\boxed{i_R = f_3(v_C, i_L, v(t))} \longrightarrow \boxed{i_R = \tfrac{1}{R}v_C}$$

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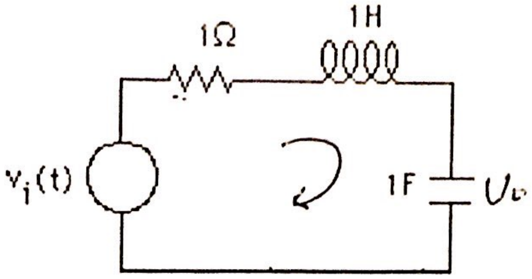
Step 4 Obtain the S-S representation in vector-matrix form

$$\begin{bmatrix} \dot{v}_C \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix} \cdot \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} v(t)$$

$$i_R = \begin{bmatrix} \frac{1}{R} & 0 \end{bmatrix} \cdot \begin{bmatrix} v_C \\ i_L \end{bmatrix}$$

Ex: Find the state equations for the electrical network in figure below



$$v_i = v_R + v_L + v_C$$

$$v_L = v_i - v_R - v_C$$

$$L\frac{di_L}{dt} = v_i - R\,i_L - v_C$$

$$\frac{di_L}{dt} = \frac{1}{L}v_i - \frac{R}{L}i_L - \frac{1}{L}v_C$$

$$x_1 = i_L$$

$$\dot{x}_1 = \frac{di_L}{dt}$$

$$x_2 = v_C$$

$$\dot{x}_2 = \frac{dv_C}{dt}$$

$$\boxed{\dot{x}_1 = -\frac{R}{L}x_1 - \frac{1}{L}x_2 + \frac{1}{L}v_i}$$

& $$C\frac{dv_C}{dt} = i_C$$ But $i_C = i_L = i_R$

$$i_L = i_C = C\frac{dv_C}{dt}$$

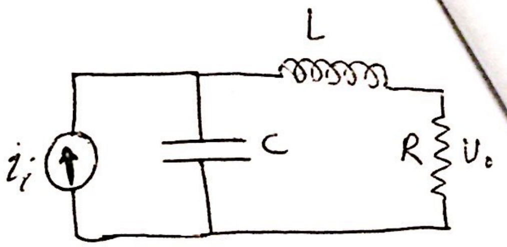
$$\frac{dv_C}{dt} = \frac{1}{C}i_L$$

$$\boxed{\dot{x}_2 = \frac{1}{C}x_1}$$

& $y = v_o = v_C = x_2 \implies y = x_2$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-\frac{R}{L} & -\frac{1}{L}\\ \frac{1}{C} & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}\frac{1}{L}\\ 0\end{bmatrix}v_i \quad \& \quad y = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

EX Find the SS Equations for the electrical network shown figure below.



$$i_i(t) = i_c(t) + i_L(t) \longrightarrow ①$$

$$v_c(t) = v_L(t) + v_R(t) \longrightarrow ②$$

eq①

$$i_c = C\frac{dv_c}{dt} = i_i - i_L$$

$$\frac{dv_c}{dt} = \frac{1}{C} i_i - \frac{1}{C} i_L$$

$$x_1 = v_c \Rightarrow \dot{x}_1 = \frac{dv_c}{dt}$$

$$x_2 = i_L \Rightarrow \dot{x}_2 = \frac{di_L}{dt}$$

$$\boxed{\dot{x}_1 = -\frac{1}{C} x_2 + \frac{1}{C} i_i}$$

eq②

$$v_L = v_c - v_R$$

$$L\frac{di_L}{dt} = v_c - i_L R$$

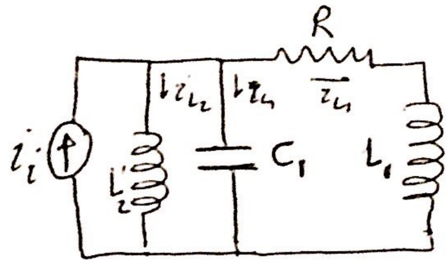
$$\frac{di_L}{dt} = \frac{1}{L} v_c - \frac{R}{L} i_L$$

$$\boxed{\dot{x}_2 = \frac{1}{L} x_1 - \frac{R}{L} x_2}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} i_i$$

$$y = v_o = R\, i_L = R x_2 \Rightarrow y = \begin{bmatrix} 0 & R \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Ex Find SS



$x_1 = i_{L_2}$
$x_2 = V_C$
$x_3 = i_{L_1}$

① $V_{C_1} = V_R + V_{L_1}$

$V_{C_1} = R i_L + V_{L_1}$

$V_{L_1} = V_{C_1} - R i_{L_1}$

$$\frac{di_{L_1}}{dt} = \frac{1}{L_1} V_{C_1} - \frac{R}{L_1} i_{L_1}$$

$$\boxed{\dot{X}_3 = \frac{1}{L_1} X_2 - \frac{R}{L_1} X_3}$$

② $v_{C_1} = v_{L_2}$

$$L_2 \frac{di_L}{dt} = v_{C_1} \Rightarrow \frac{di_{L_2}}{dt} = \frac{1}{L_2} V_{C_1}$$

$$\boxed{\dot{X}_1 = \frac{1}{L_2} X_2}$$

③ $i_i = i_{C_1} + i_{L_1} + i_{L_2}$

$i_{C_1} = i_i - i_{L_1} - i_{L_2}$

$$C\frac{dv_{C_1}}{dt} = i_i - i_{L_1} - i_{L_2} \Rightarrow \frac{dv_{C_1}}{dt} = \frac{1}{C} i_i - \frac{1}{C} i_{L_1} - \frac{1}{C} i_{L_2}$$

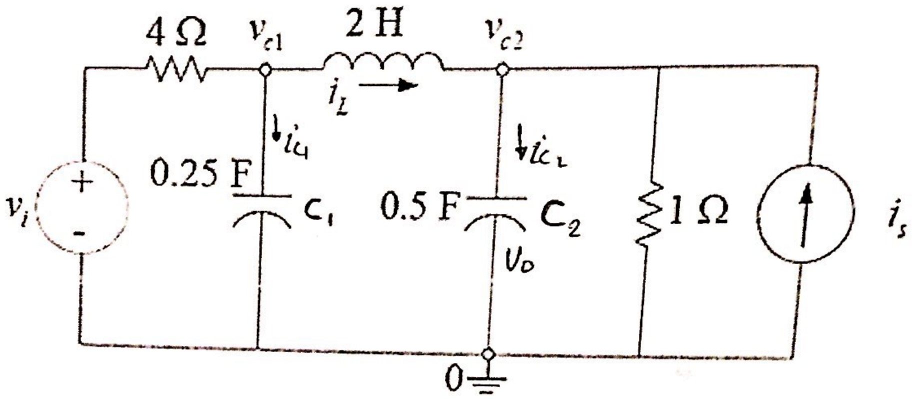
$$\boxed{\dot{X}_2 = -\frac{1}{C} X_3 - \frac{1}{C} X_1 - \frac{1}{C} i_i}$$

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{L_2} & 0 \\ -\frac{1}{C} & 0 & -\frac{1}{C} \\ 0 & \frac{1}{L_1} & -\frac{R}{L_1} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{C} \\ 0 \end{bmatrix} i_i$$

$$y = e_0 = V_{L_1} = V_{C_1} - V_R = V_{C_1} - R i_{L_1} \Rightarrow y = X_2 - R X_3$$

$$y = \begin{bmatrix} 0 & 1 & -R \end{bmatrix} \begin{bmatrix} X_1 \\ X_1 \\ X_3 \end{bmatrix}$$

Ex Write the state equation for the network



Define the state variables as current through the inductor and voltage across the capacitors. Write two node equations containing capacitors and a loop equation containing the inductor. The state variables are v_{c1}, v_{c_2}, and i_L.

Node equations are

$$\frac{V_i - V_{c1}}{4} = i_{c1} + i_L$$

$$0.25\frac{dv_{c1}}{dt} + i_L + \frac{v_{c1} - v_i}{4} = 0 \quad \Rightarrow \quad \dot{v}_c = -v_{c1} - 4i_L + v_i$$

$$i_i - \frac{V_{c2}}{1} = i_{c2} - i_L$$

$$0.5\frac{dv_{c2}}{dt} - i_L + \frac{v_{c2}}{1} - i_s = 0 \quad \Rightarrow \quad \dot{v}_{c2} = -2v_{c1} + 2v_{c2} + 2i_s$$

and the loop equation is

$$2\frac{di_L}{dt} + v_{c2} - v_{c1} = 0 \quad \Rightarrow \quad \dot{i}_L = 0.5v_{c1} - 0.5v_{c2}$$

or

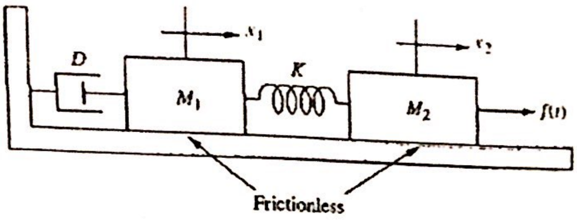
$$\begin{bmatrix} \dot{v}_{c1} \\ \dot{v}_{c2} \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} -1 & 0 & -4 \\ 0 & -2 & 2 \\ 0.5 & -0.5 & 0 \end{bmatrix} \begin{bmatrix} v_{c1} \\ v_{c2} \\ i_L \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_i \\ i_s \end{bmatrix}$$

$$y = V_o = V_{c2}$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} V_{c1} \\ V_{c2} \\ i_L \end{bmatrix}$$

System
2- Input
3- State
1- output.

Ex: Find the state equations for the translational mechanical system in figure below



$$M_1\frac{d^2x_1}{dt^2}+D\frac{dx_1}{dt}+Kx_1-Kx_2=0$$

$$-Kx_1+M_2\frac{d^2x_2}{dt^2}+Kx_2=f(t)$$

$$\frac{dx_1}{dt}= \qquad +v_1$$

$$\frac{dv_1}{dt}=-\frac{K}{M_1}x_1-\frac{D}{M_1}v_1+\frac{K}{M_1}x_2$$

$$\frac{dx_2}{dt}= \qquad +v_2$$

$$\frac{dv_2}{dt}=+\frac{K}{M_2}x_1 \qquad -\frac{K}{M_2}x_2 \qquad +\frac{1}{M_2}f(t)$$

In vector-matrix form,

$$\begin{bmatrix}\dot{x}_1\\ \dot{v}_1\\ \dot{x}_2\\ \dot{v}_2\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & 0\\ -K/M_1 & -D/M_1 & K/M_1 & 0\\ 0 & 0 & 0 & 1\\ K/M_2 & 0 & -K/M_2 & 0\end{bmatrix}\begin{bmatrix}x_1\\ v_1\\ x_2\\ v_2\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 0\\ 1/M_2\end{bmatrix}f(t)$$

or

$$Z_1 = X_1 \longrightarrow \dot{Z}_1 = \dot{X}_1 = Z_2$$

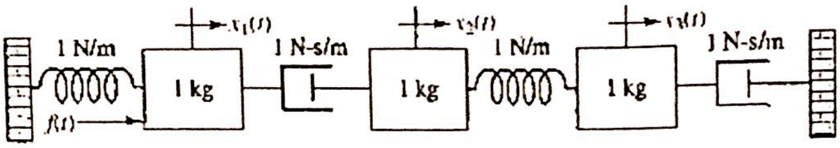
$$Z_2 = \dot{X}_1 \longrightarrow \dot{Z}_2 = \ddot{X}_1 = -\frac{D}{M_1}Z_2 - \frac{K}{M_1}Z_1 + \frac{K}{M_1}Z_3$$

$$Z_3 = X_2 \longrightarrow \dot{Z}_3 = \dot{X}_2 = Z_4$$

$$Z_4 = \dot{X}_2 \longrightarrow Z_4 = \ddot{X}_1 = \frac{f(t)}{M_2} - \frac{K}{M_2}Z_3 + \frac{K}{M_2}Z_1$$

$$\begin{bmatrix}\dot{Z}_1\\ \dot{Z}_2\\ \dot{Z}_3\\ \dot{Z}_4\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & 0\\ -\frac{K}{M_1} & -\frac{D}{M_1} & \frac{K}{M_1} & 0\\ 0 & 0 & 0 & 1\\ \frac{K}{M_2} & 0 & -\frac{K}{M_2} & 0\end{bmatrix}\begin{bmatrix}Z_1\\ Z_2\\ Z_3\\ Z_4\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 0\\ 1/M_2\end{bmatrix}f(t)$$

EX: Represent the following transfer function in state space. Give your answer in ve matrix form.



Writing the equations of motion

$$\begin{aligned}
(s^2+s+1)X_1(s) \quad & -sX_2(s) & & = F(s) \\
-sX_1(s) + (s^2+s+1)X_2(s) & & -X_3(s) & = 0 \\
& -X_2(s) + (s^2+s+1)X_3(s) & & = 0
\end{aligned}$$

Taking the inverse Laplace transform and simplifying,

$$\ddot{x}_1 = -\dot{x}_1 - x_1 + \dot{x}_2 + f$$

$$\ddot{x}_2 = \dot{x}_1 - \dot{x}_2 - x_2 + x_3$$

$$\ddot{x}_3 = -\dot{x}_3 - x_3 + x_2$$

Defining state variables, z_i,

$$z_1 = x_1;\ z_2 = \dot{x}_1;\ z_3 = x_2;\ z_4 = \dot{x}_2;\ z_5 = x_3;\ z_6 = \dot{x}_3$$

Writing the state equations using the definition of the state variables and the inverse transform of the differential equation,

$$\dot{z}_1 = z_2$$

$$\dot{z}_2 = \ddot{x}_1 = -\dot{x}_1 - x_1 + \dot{x}_2 + f = -z_2 - z_1 + z_4 + f$$

$$\dot{z}_3 = \dot{x}_2 = z_4$$

$$\dot{z}_4 = \ddot{x}_2 = \dot{x}_1 - \dot{x}_2 - x_2 + x_3 = z_2 - z_4 - z_3 + z_5$$

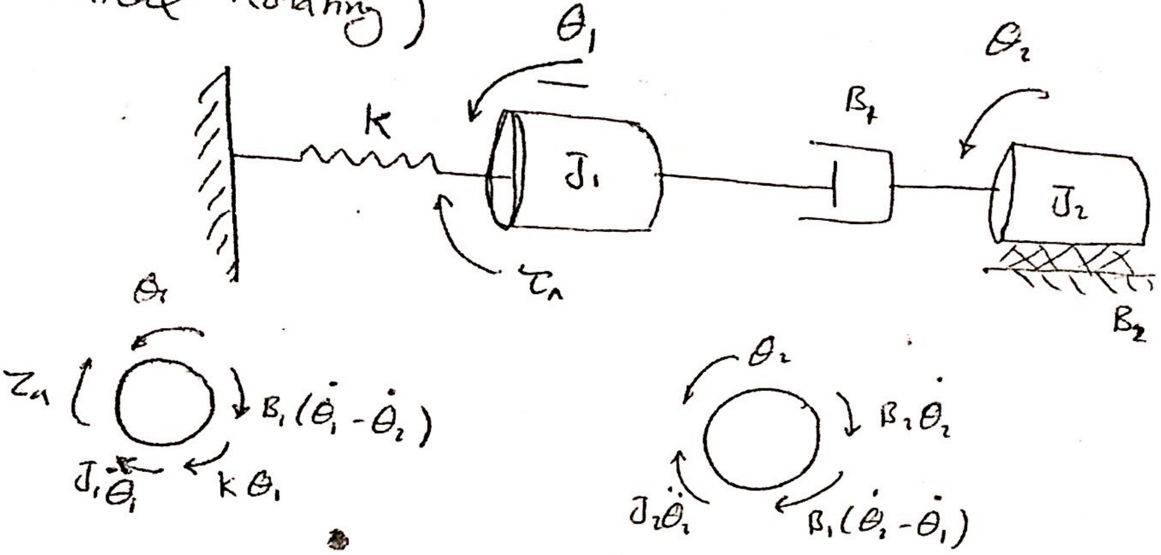
$$\dot{z}_5 = \dot{x}_3 = z_6$$

$$\dot{z}_6 = \ddot{x}_3 = -\dot{x}_3 - x_3 + x_2 = -z_6 - z_5 + z_3$$

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} f(t)$$

$$y = [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0]\mathbf{z}$$

EX: Mechanical Rotating)



$$J_1\ddot\theta_1 + B_{r1}(\dot\theta_1 - \dot\theta_2) + k\theta_1 = -\tau_a$$

$$J_2\ddot\theta_2 + B_f(\dot\theta_2 - \dot\theta_1) + B_2\dot\theta_2 = 0$$

$$\dot X_1 = X_2$$

$$\dot X_2 = -\frac{\tau_a}{J} - \frac{B_{r1}}{J}X_2 + \frac{B_{r1}}{J}X_3 - \frac{K}{J}X_1$$

$$\dot X_3 = -\frac{B_1}{J_2}X_3 + \frac{B_1}{J_2}X_2 - \frac{B_2}{J_2}X_3$$

لاحظ لا يوجد θ_2

$x_1 = \theta_1 \longrightarrow \dot x_1 = x_2$

$x_2 = \dot\theta_1 \quad \dot x_2 = \ddot\theta_1 = -$

$x_3 = \dot\theta_2 \quad \dot x_3 = \theta_2 = -$

$$\begin{bmatrix}\dot X_1\\ \dot X_2\\ \dot X_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ -\frac{K}{J} & -\frac{B_{r1}}{J} & \frac{B_{r1}}{J}\\ 0 & \frac{B_1}{J_2} & -\frac{(B_2+B_1)}{J}\end{bmatrix}\begin{bmatrix}X_1\\ X_2\\ X_3\end{bmatrix} + \begin{bmatrix}0\\ -\frac{1}{J}\\ 0\end{bmatrix}\tau_a$$

$$y = \theta_1 = x_1$$

$$y = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

Converting from the Differential Equation to State Space

Ex. Obtain the state equation in phase variable form for the following differential equation.

$$\frac{d^3y}{dt^3}+2\frac{d^2y}{dt^2}+3.\frac{dy}{dt}+4y= 5u(t)$$

The differential equation is third order, thus there are three state variables as follows

$$x_1 = y, \quad x_2 = \dot{y}, \quad x_3 = \ddot{y}$$

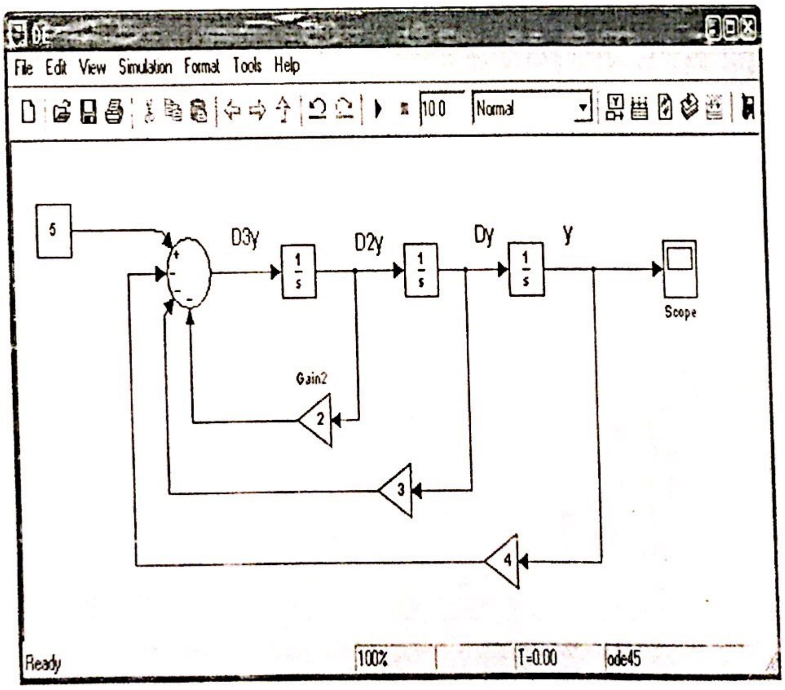
and the derivatives are

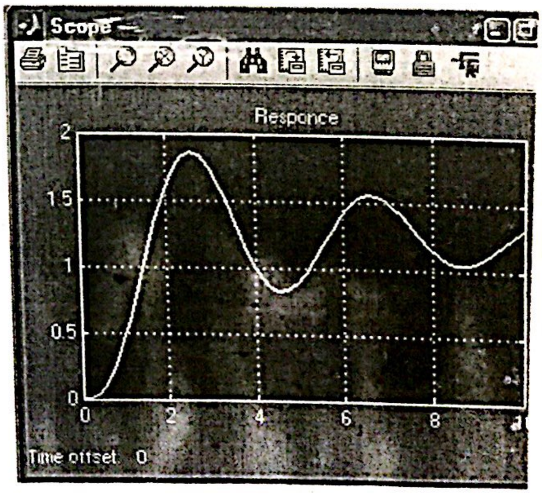
$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \text{ and } \dot{x}_3 = -4x_1 - 3x_2 - 2x_3 + 5u(t)$$

Or in matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} 5u(t)$$

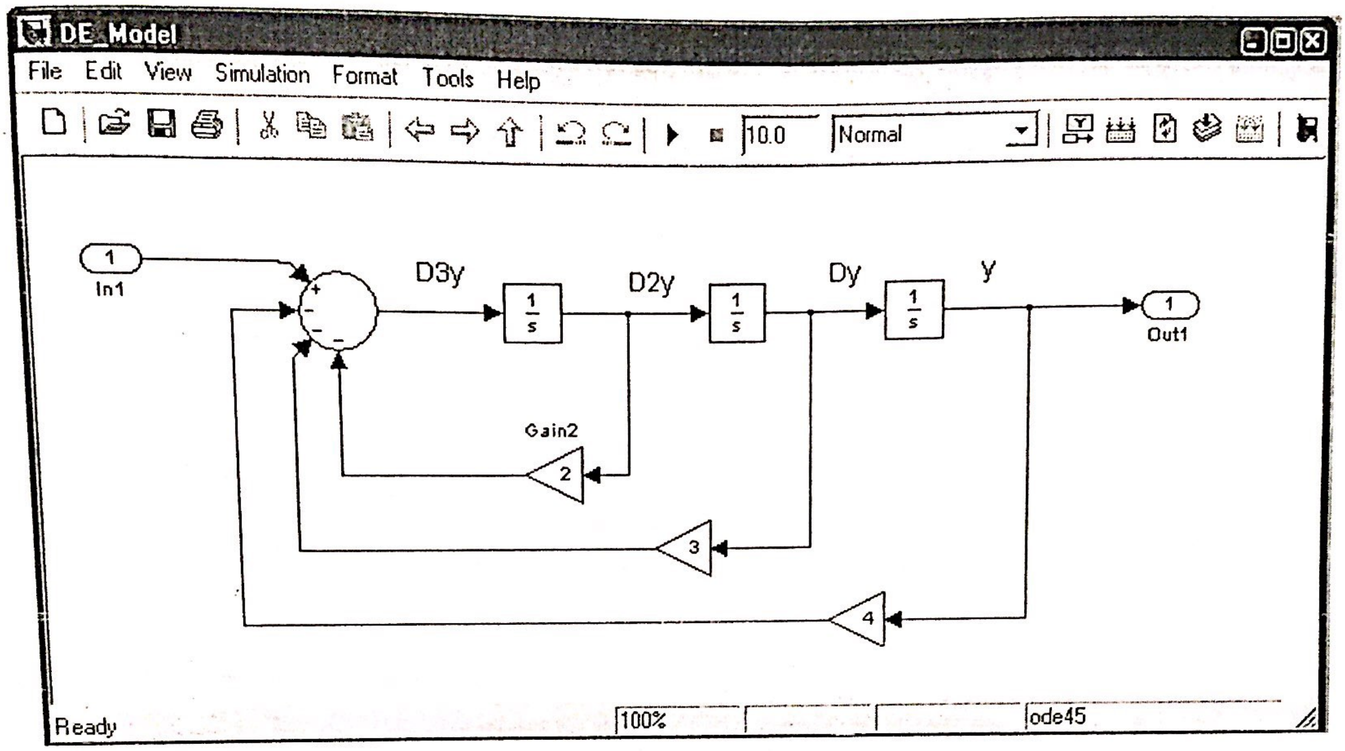
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$





Extract the state space using simulink]

1- Save the differential equation in the following model



2- in the command windows type the following command

```
>> [A,B,C,D]=linmod('DE_model')

A =
     0     1     0
     0     0     1
    -4    -3    -2

B =
     0
     0
     1

C =
     1     0     0

D =
     0
```

Converting from a Transfer Function to State Space

1- Converting a transfer function with constant term in numerator

This case corresponds to a linear system that can be represented as an n^th^ -order linear differential equation with constant coefficients like:

$$\boxed{\frac{d^n y(t)}{dt^n}+a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}}+\dots a_1\frac{dy(t)}{dt}+a_0 y(t)=b_0 u(t)}$$

The classical Transfer Function (TF) representation of that system is obtained applying the Laplace Transform to the diff. eq.

$$\boxed{G(s)=\frac{Y(s)}{U(s)}=\frac{b_0}{s^n+a_{n-1}s^{n-1}+\dots a_1 s+a_0}}$$

$$U(s)\longrightarrow\boxed{G(s)}\longrightarrow Y(s)$$

- If the State Space (SS) representation has to be obtained, a convenient way to select state variables is to choose the output, y(t), and its n-1 derivatives as the state variables. This is called the **phase-variable** choice.
  A SS representation using the phase-variable choice for the state variables is said to be in the Controller Canonical Form (CCF).

$$\boxed{\begin{aligned}x_1&=y\\x_2&=\frac{dy}{dt}\\x_3&=\frac{d^2y}{dt^2}\\&\vdots\\x_n&=\frac{d^{n-1}y}{dt^{n-1}}\end{aligned}}\xrightarrow{\text{Differentiating both sides}}\boxed{\begin{aligned}\dot x_1&=\frac{dy}{dt}\\\dot x_2&=\frac{d^2y}{dt^2}\\\dot x_3&=\frac{d^3y}{dt^3}\\&\vdots\\\dot x_n&=\frac{d^n y}{dt^n}\end{aligned}}\xrightarrow{\text{Substituting}}\boxed{\begin{aligned}\dot x_1&=x_2\\\dot x_2&=x_3\\&\vdots\\\dot x_{n-1}&=x_n\\\dot x_n&=-a_0x_1-a_1x_2\dots-a_{n-1}x_n+b_0u\end{aligned}}$$

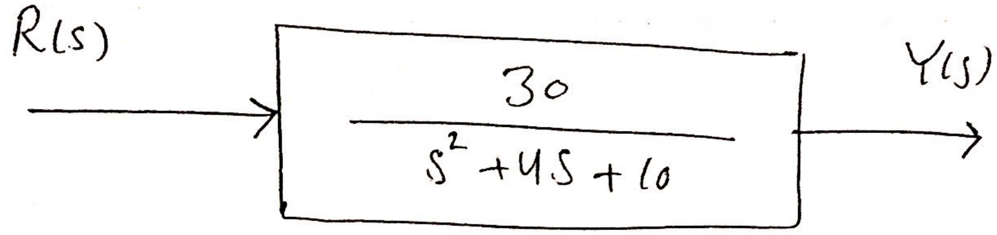
that can be written in vector-matrix form as follows:

$$\begin{bmatrix}\dot x_1\\\dot x_2\\\vdots\\\dot x_{n-1}\\\dot x_n\end{bmatrix}=\underbrace{\begin{bmatrix}0&1&0&\dots&0\\0&0&1&\dots&0\\\vdots&&&&\\0&0&0&\dots&1\\-a_0&-a_1&-a_2&\dots&-a_{n-1}\end{bmatrix}}_{A_{CCF}}\begin{bmatrix}x_1\\x_2\\\vdots\\x_{n-1}\\x_n\end{bmatrix}+\underbrace{\begin{bmatrix}0\\0\\\vdots\\0\\b_0\end{bmatrix}}_{B_{CCF}}u$$

$$y=\underbrace{\begin{bmatrix}1&0&0&\dots&0\end{bmatrix}}_{C_{CCF}}\cdot\begin{bmatrix}x_1\\x_2\\\vdots\\x_{n-1}\\x_n\end{bmatrix}$$

Q- What is mean by the Phase-variable form of the state equation?
The state variable are successive derivatives.

EX Represent the following transfer function in state space



$$\frac{Y(s)}{R(s)} = \frac{30}{s^2+4s+10}$$

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 10\,y(t) = 30\,r(t)$$

$$x_1 = y \longrightarrow \dot{x}_1 = \dot{y} = x_2$$

$$x_2 = \dot{y} \longrightarrow \dot{x}_2 = \ddot{y} = 30\,r(t) - 4x_2 - 10x_1$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -10x_1 - 4x_2 + 30\,r(t)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 30 \end{bmatrix} r(t)$$

&

$$y = x_1$$

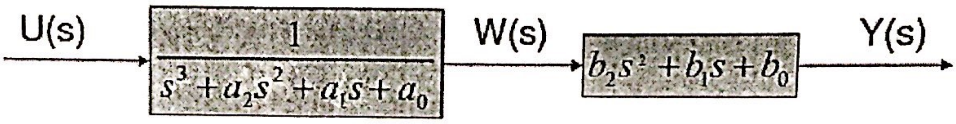
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

2- Converting a transfer function with polynomial in numerator

For example, consider a third order SISO system with second-order numerator polynomial

$$\boxed{G(s)=\frac{b_2s^2+b_1s+b_0}{s^3+a_2s^2+a_1s+a_0}}$$

First, separate the transfer function into two cascaded modules



We can write:

$$\boxed{W(s)=\frac{1}{s^3+a_2s^2+a_1s+a_0}U(s)} \xrightarrow{\text{In the time domain}} \boxed{\dddot{w}(t)+a_2\ddot{w}(t)+a_1\dot{w}(t)+a_0w(t)=u(t)}$$

$$\boxed{Y(s)=\left(b_2s^2+b_1s+b_0\right)\cdot W(s)} \qquad \boxed{y(t)=b_2\ddot{w}(t)+b_1\dot{w}(t)+b_0w(t)}$$

$$\boxed{\begin{aligned}x_1&=w\\x_2&=\dot{w}=\dot{x}_1\\x_3&=\ddot{w}=\ddot{x}_1=\dot{x}_2\end{aligned}}$$

$$\boxed{\dot{x}_3=-a_2x_3-a_1x_2-a_0x_1+u}$$

$$\boxed{\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\-a_0&-a_1&-a_2\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}u(t)} \quad \text{State eq}$$

$$\boxed{w(t)=\begin{bmatrix}1&0&0\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}0\end{bmatrix}\cdot u(t)} \quad \text{Intermediate eq}$$

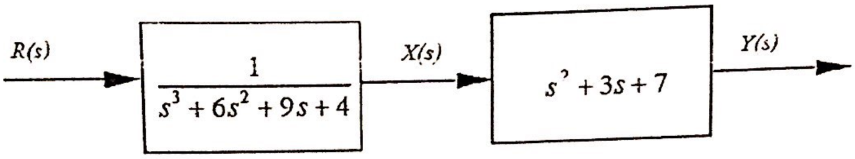
$$\boxed{\begin{aligned}y(t)&=b_2\ddot{w}(t)+b_1\dot{w}(t)+b_0w(t)\\&=b_2x_3+b_1x_2+b_0x_1\end{aligned}}$$

$$\boxed{y(t)=\begin{bmatrix}b_0&b_1&b_2\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}0\end{bmatrix}\cdot u(t)} \quad \text{Output eq}$$

EX: Represent the following transfer function in state space. Give your answer in vector-matrix form.

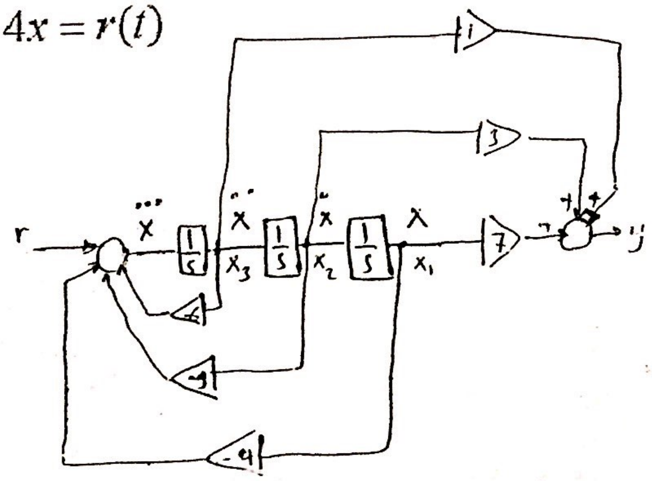
$$\xrightarrow{\quad} \boxed{\frac{(s^2+3s+7)}{(s+1)(s^2+5s+4)}} \xrightarrow{\quad}$$

The transfer function can be represented as a block diagram as follows:



Writing the differential equation for the first box:

$$\dddot{x}+6\ddot{x}+9\dot{x}+4x=r(t)$$



Block diagram

Defining the state variables:

$$x_1 = x$$

$$x_2 = \dot{x}$$

Thus

$$x_3 = \ddot{x}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -4x-9\dot{x}-6\ddot{x}+r(t) = -4x_1-9x_2-6x_3+r(t)$$

$$y=\ddot{x}+3\dot{x}+7x=7x_1+3x_2+x_3$$

From the second box,

In vector-matrix form:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -9 & -6 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r(t)$$

$$y = \begin{bmatrix} 7 & 3 & 1 \end{bmatrix}\mathbf{x}$$

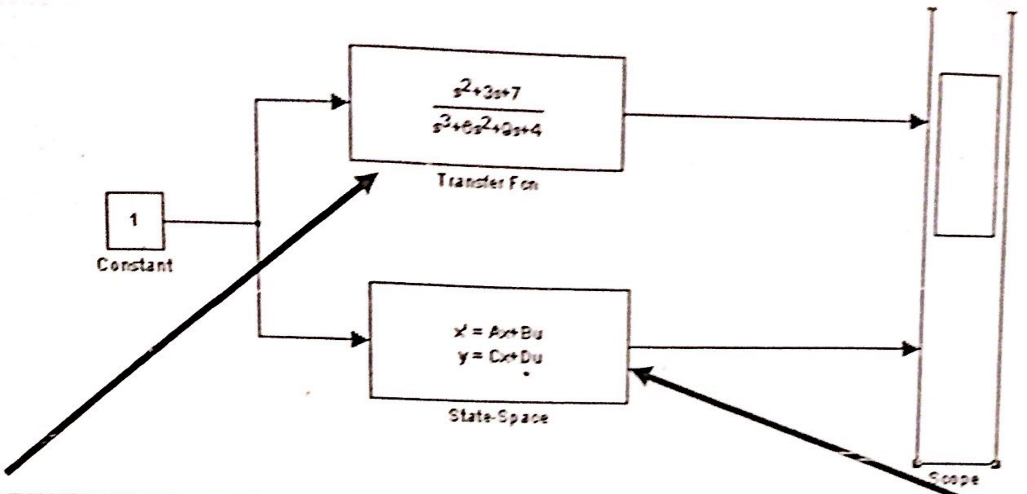
MATLAB Code

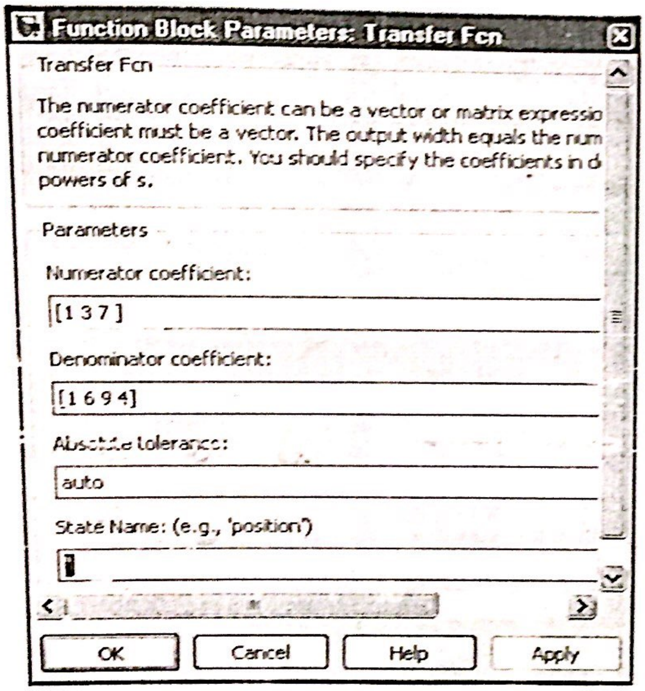
```
clc
num=[1 3 7];
den=conv([ 1 1],[ 1 5 4]);
G=tf(num,den)
[A,B,C,D]=tf2ss(num,den)

' convert to canonical form'
Af=flipud(A);
Af=fliplr(Af)
Bf=flipud(B)
Cf=fliplr(C)

'statespace'
statespace = ss(Af,Bf,Cf,D)
```

```
Transfer function:
    s^2 + 3 s + 7
---------------------
s^3 + 6 s^2 + 9 s + 4


A =

    -6    -9    -4
     1     0     0
     0     1     0

B =

C =

     1     3     7

D =
```

```
convert to canonical form


Af =

     0     1     0
     1     0     0
    -6    -9    -4


Af =

     0     1     0
     0     0     1
    -4    -9    -6


Bf =

     0
     0
     1


Cf =

     7     3     1
```

```
statespace


a =
            x1   x2   x3
       x1    0    1    0
       x2    0    0    1
       x3   -4   -9   -6

b =
           u1
       x1   0
       x2   0
       x3   1

c =
           x1   x2   x3
       y1   7    3    1

d =
           u1
       y1   0
```





Transfer Fcn

The numerator coefficient can be a vector or matrix expressio
coefficient must be a vector. The output width equals the num
numerator coefficient. You should specify the coefficients in d
powers of s.

Parameters

Numerator coefficient:

[1 3 7]

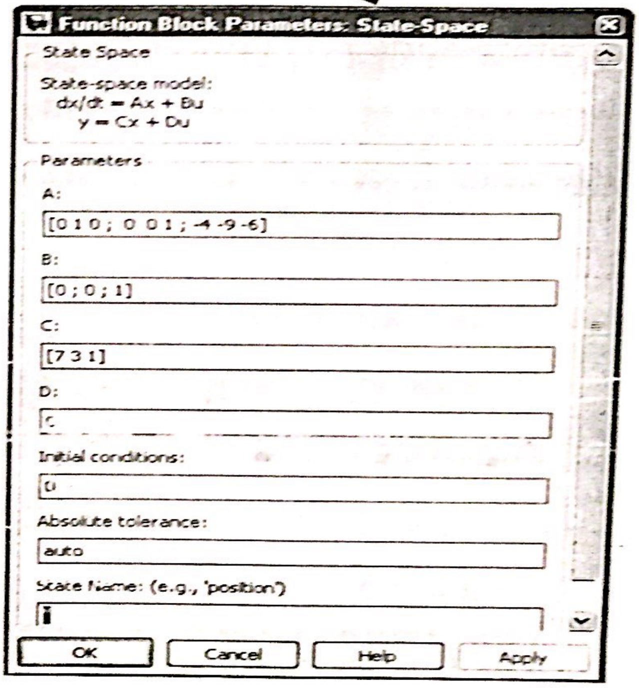
Denominator coefficient:

[1 6 9 4]

Absolute tolerance:

auto

State Name: (e.g., 'position')



State Space

State-space model:
dx/dt = Ax + Bu
y = Cx + Du

Parameters

A:

[0 1 0 ; 0 0 1 ; -4 -9 -6]

B:

[0 ; 0 ; 1]

C:

[7 3 1]

D:

0

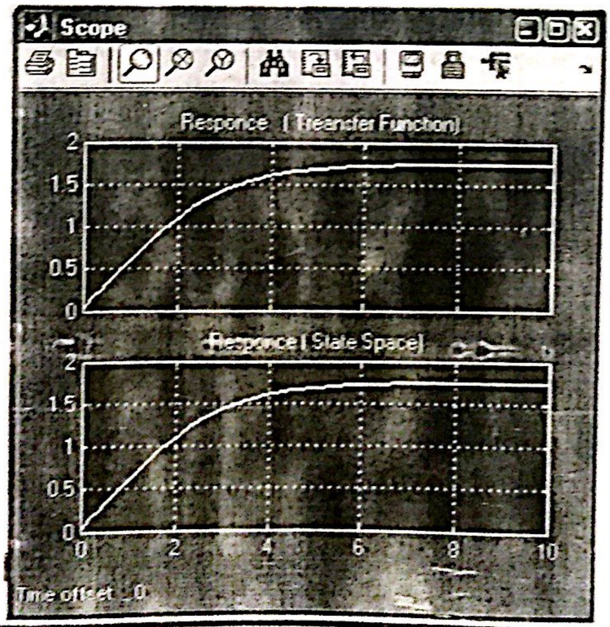
Initial conditions:

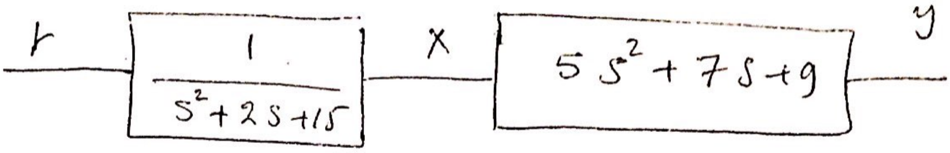
0

Absolute tolerance:

auto

State Name: (e.g., 'position')





$$\ddot{x} + 2\dot{x} + 15x = r$$

$$x_1 = x \quad \Rightarrow \dot{x}_1 = x_2$$

$$x_2 = \dot{x} \quad \Rightarrow \dot{x}_2 = r - 15x_1 - 2x_2$$

&

$$y = 5\ddot{x} + 7\dot{x} + 9x$$

$$= 5(r - 15x - 2\dot{x}) + 7\dot{x} + 9x$$

$$= 5r - 75x - 10\dot{x} + 7\dot{x} + 9x$$

$$= 5r - 66x - 3\dot{x}.$$

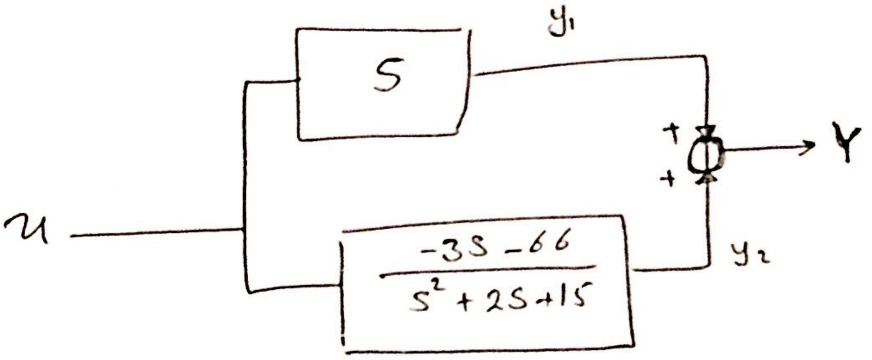
$$\therefore \; y = 5r - 66x_1 - 3x_2$$

$$= \begin{bmatrix} -66 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 5 \cdot r$$

Find the SS

$$G(s) = \frac{5s^2+7s+9}{s^2+2s+15}$$

$$= 5 + \frac{-3s-66}{s^2+2s+15}$$



$$\frac{y_2}{u} = \frac{-3s-66}{s^2+2s+15}$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -15 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}$$

$$y_2 = \begin{bmatrix}-66 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + 0 \cdot u$$

$$y_1 = 5u$$

$$y = Y = y_1 + y_2$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -15 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$

$$y = \begin{bmatrix}-66 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \underset{D}{5}\,u$$

Converting from a State Space to a Transfer Function

Given the state and output equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

take the Laplace transform assuming zero initial conditions:

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$
$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)$$

Solving for $\mathbf{X}(s)$ in Eq.

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}\mathbf{U}(s)$$

or

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

where **I** is the identity matrix.
Substituting Eq. ( ) into Eq. ( ) yields

$$\mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\mathbf{U}(s)$$

We call the matrix $[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]$ the transfer function matrix, since it relates

$$\boxed{T(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D}}$$

Ex Convert the state space and output equations to a transfer function

- State space model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{[0]}_{D} u$$

- Using the expression for derived transfer function

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2+3s+2} = \frac{1}{(s+1)(s+2)}$$

$$T(s) = C(sI-A)^{-1}B+D$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + [0]$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\frac{1}{s(s+3)+2} \right) \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^2+3s+2} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ s \end{bmatrix} = \frac{1}{s^2+3s+2}$$

Ex: Convert the state space and output equations to a transfer function

$$\mathbf{x} = \begin{bmatrix} -4 & -1.5 \\ 4 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}u(t)$$

$$y = \begin{bmatrix} 1.5 & 0.625 \end{bmatrix}\mathbf{x}$$

The state equation is converted to a transfer function using

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \qquad (1)$$

where

$$\mathbf{A} = \begin{bmatrix} -4 & -1.5 \\ 4 & 0 \end{bmatrix}, \; \mathbf{B} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \text{ and } \mathbf{C} = \begin{bmatrix} 1.5 & 0.625 \end{bmatrix}$$

Evaluating $(s\mathbf{I} - \mathbf{A})$ yields

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s+4 & 1.5 \\ -4 & s \end{bmatrix}$$

Taking the inverse we obtain

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s^2 + 4s + 6}\begin{bmatrix} s & -1.5 \\ 4 & s+4 \end{bmatrix}$$

Substituting all expressions into Eq. (1) yields

$$G(s) = \frac{3s + 5}{s^2 + 4s + 6}$$

MATLAB code

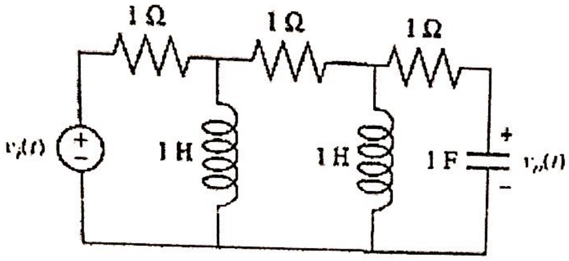
```
A=[ -4 -1.5 ; 4 0 ];
B=[ 2 ; 0 ];
C=[1.5 0.625];
D=0;
[num,den]= ss2tf(A,B,C,D);
G= tf(num,den)
```

```
Transfer function:
   3 s + 5
-------------
s^2 + 4 s + 6
```

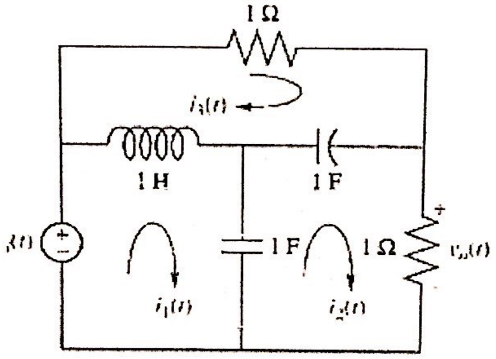
Home Work 3 -------------------EE362

①Find the state-space representation of the following systems ② Convert SS to T.F using MATLAB

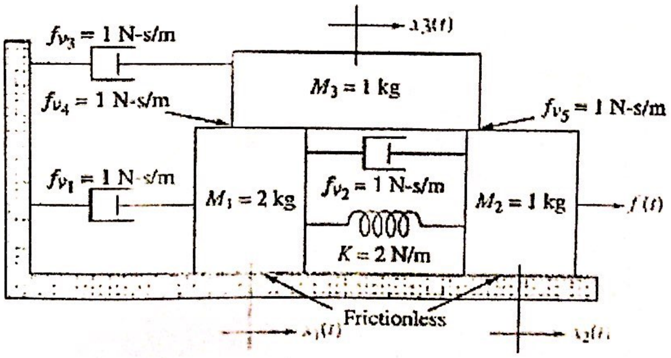
Case 1



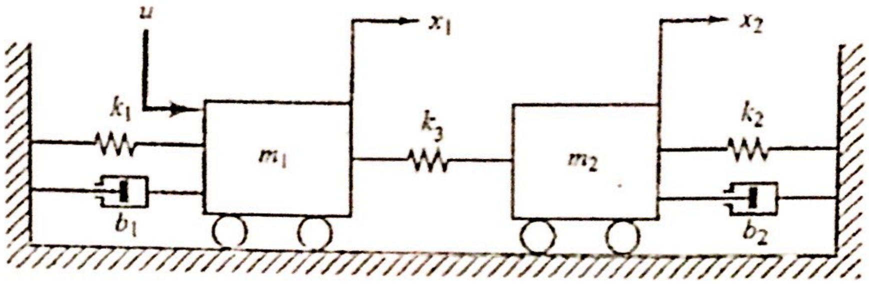
Case 2



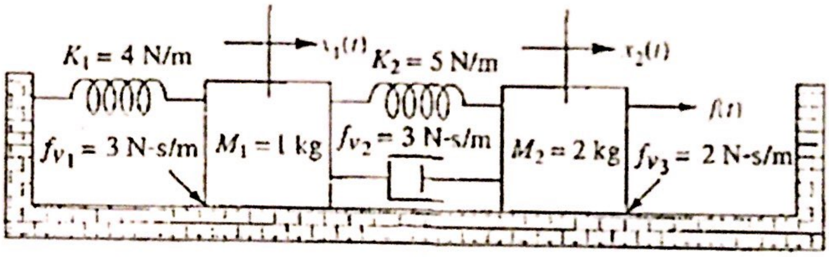
Case 3



Case 4



Case 5



Case 6